偏微分方程式と差分法

(Partial Differential Equations and Finite Difference Method)

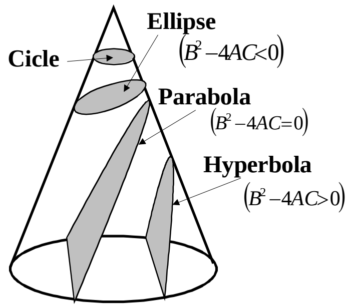
1. 2階の偏微分方程式の区分 (Clarification of partial differential equations with 2nd order)

1.1 円錐曲線（二次曲線）(Conical curve geometry and quadratic curve on the section of conical)

$$u = u(x, y) \tag{1}$$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey \quad = G \tag{2}$$

ⅰ) $B^2 - 4AC < 0$: 楕円（円を含む）(Ellipse)



Conic Sections

ⅱ) $B^2 - 4AC = 0$: 放物線(Parabola)

ⅲ) $B^2 - 4AC > 0$: 双曲線(Hyperbola)

1.2 偏微分方程式の種類（***Partial Differential Equations***）

$$u = u(x, y),\ u_x = \frac{\partial u}{\partial x},\ u_y = \frac{\partial u}{\partial y},\ u_{xx} = \frac{\partial^2 u}{\partial x^2},\ u_{yy} = \frac{\partial^2 u}{\partial y^2},\ u_{xy} = \frac{\partial^2 u}{\partial x\, \partial y} \tag{3}$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{4}$$

ⅰ) $B^2 - 4AC < 0$: 楕円型方程式: 定常状態の現象（steady state）

Laplace Eq. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (u_{xx} + u_{yy} = 0) \tag{5}$$

Poisson Eq. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \quad (u_{xx} + u_{yy} = f) \tag{6}$$

ⅱ) $B^2 - 4AC = 0$: 放物型方程式: 熱流(heat flux)、拡散(diffusion)、圧密(consolidation)

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad (u_t = \alpha^2 u_{xx}) \tag{7}$$

ⅲ) $B^2 - 4AC > 0$: 双曲型方程式: 振動（undulation, vibration）、波動（Wave propagation）

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad (u_{tt} = \alpha^2 u_{xx}) \tag{8}$$

2. 2階の偏微分方程式の差分化 (Finite difference equation of differential equations with 2nd order)

2.1 放物型(Parabola type): 熱伝導方程式の差分化（finite difference equation of thermal conduction）

Taylor's expansion of a temperature $T(t, x)$ as function of time t and location x.

$$T(t, x+\Delta x) = T(t, x) + \frac{\partial T}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial T}{\partial x}(\Delta x)^2 + \frac{1}{3!}\frac{\partial T}{\partial x}(\Delta x)^3 + O\left((\Delta x)^4\right) \tag{9}$$

$$T(t, x-\Delta x) = T(t, x) - \frac{\partial T}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial T}{\partial x}(\Delta x)^2 - \frac{1}{3!}\frac{\partial T}{\partial x}(\Delta x)^3 + O\left((\Delta x)^4\right) \tag{10}$$

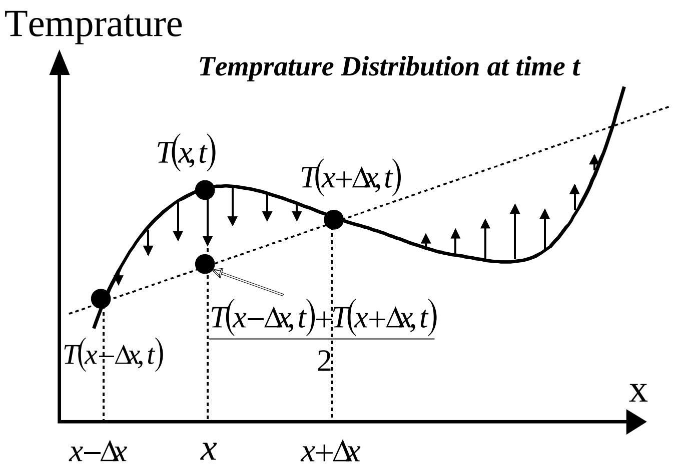
By eq.(9) and eq.(10),

From (9)-(10): $$T_x = \frac{\partial T}{\partial x} \approx \frac{1}{2\Delta x}\left[T(x+\Delta x, t) - T(x-\Delta x, t)\right] \tag{11}$$

From (9)+(10): $$T_{xx} = \frac{\partial^2 T}{\partial x^2} \approx \frac{1}{\Delta x^2}\left[T(x+\Delta x, t) + T(x-\Delta x, t) - 2T(x, t)\right] = \frac{2}{(\Delta x)^2}\left[\frac{T(x+\Delta x, t) + T(x-\Delta x, t)}{2} - T(x, t)\right]$$

$$\frac{\partial T}{\partial t} = \alpha^2\frac{\partial^2 T}{\partial x^2} \approx \frac{2}{(\Delta x)^2}\left[\frac{T(x+\Delta x, t) + T(x-\Delta x, t)}{2} - T(x, t)\right] \tag{12}$$

$$\frac{\partial T}{\partial t} \propto \left[\left[\frac{T(x+\Delta x, t) + T(x-\Delta x, t)}{2} - T(x, t)\right]\right] \tag{13}$$



T_{xx} means curvature of temperature distribution.

- $T(x,t)$が近傍点の温度の平均より小さい、$T_{xx} \geq 0$：xから熱流量が流入(inflow)

  When T_x is lower then that around x, T_{xx} >0 and heat flux flow in region x.

- $T(x,t)$が近傍点の温度の平均に等しい、$T_{xx} = 0$：xへ熱流量なし(no flow)

  When T_x equals to that around x, T_{xx} =0 and no heat flux generates.

- $T(x,t)$が近傍点の温度の平均より大きい、$T_{xx} \leq 0$：xから熱流量が流出(outflow)

  When T_x is higher then that around x, T_{xx} <0 and heat flux flow out of region x.

xの熱変化速度$\dfrac{\partial T}{\partial t}$は近傍点の温度の平均とその点$x$との温度差に比例し、その増減は差の符号に従う。

- 温度$T(x,t)$が近傍点の温度の平均より小さければ、xの温度は上昇

  When T_x is lower then that around x, temperature T_x in region x increases.

- 温度$T(x,t)$が近傍点の温度の平均に等しいければ、xの温度変化なし

  When T_x eauals to that at x, temperature T_x in region x does not change.

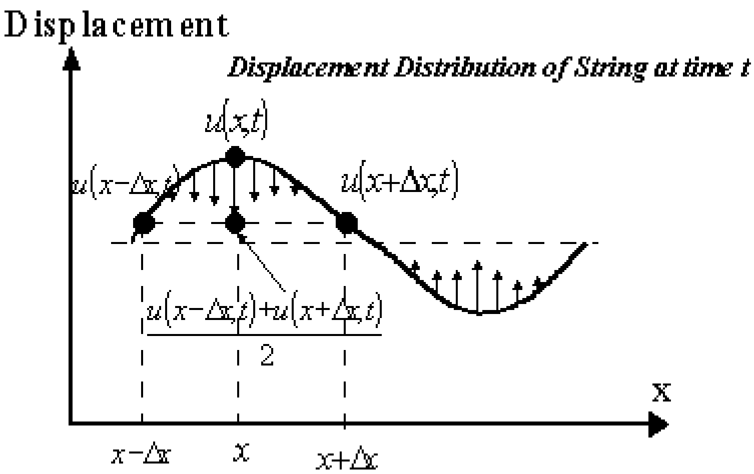
- 温度$T(x,t)$が近傍点の温度の平均より大きければ、xの温度は下降

  When T_x is higher then that around x, temperature T_x in region x decreases

2.2 双曲型放物型(hyperbola type): 波動方程式の差分化（finite difference equation of wave propagation）

$$\frac{\partial u}{\partial x} \approx \frac{1}{2\Delta x}\left[u(x+\Delta x,t)-u(x-\Delta x,t)\right] \tag{14}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &\approx \frac{1}{\Delta x^2}\left[u(x+\Delta x,t)+u(x-\Delta x,t)-2u(x,t)\right] \\ &= \frac{2}{(\Delta x)^2}\left[\frac{u(x+\Delta x,t)+u(x-\Delta x,t)}{2}-u(x,t)\right]\end{aligned} \tag{15}$$



- 変位$u(x,t)$が近傍点の変位の平均より小さければ、$u_{xx} \geq 0$: 上向き復元力
- 変位$u(x,t)$が近傍点の変位の平均に等しいければ、$u_{xx} = 0$: 復元力なし
- 変位$u(x,t)$が近傍点の変位の平均より大きければ、$u_{xx} \leq 0$: 下向き復元力

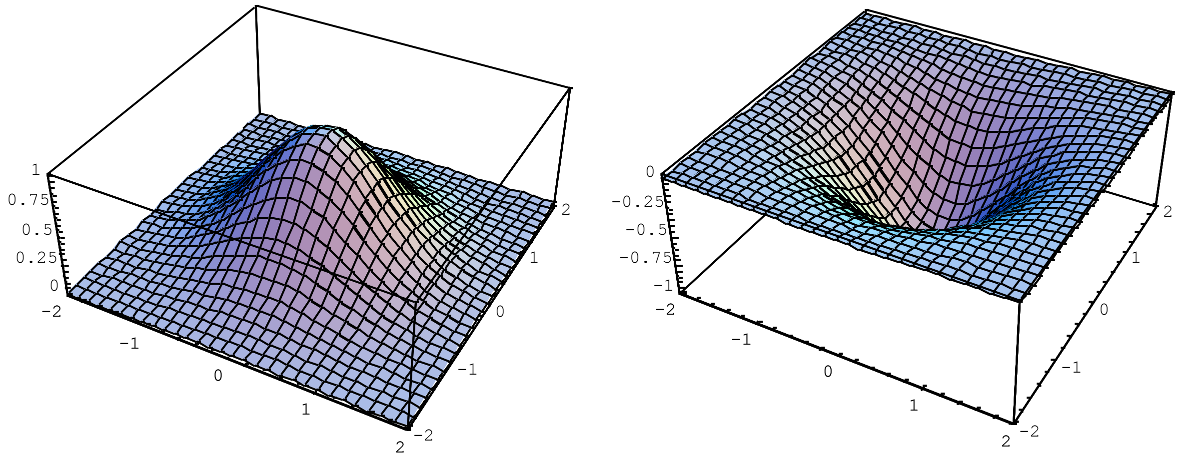
$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \approx \frac{2}{(\Delta x)^2}\left[\frac{u(x+\Delta x,t)+u(x-\Delta x,t)}{2}-u(x,t)\right] \tag{16}$$

$$\frac{\partial^2 u}{\partial t^2} \propto \left[\!\left[\frac{u(x+\Delta x,t)+u(x-\Delta x,t)}{2}-u(x,t)\right]\!\right] \tag{17}$$

xの加速度$\frac{\partial^2 u}{\partial t^2}$は近傍点の変位の平均とその点$x$との変位との差に比例し、その加減は差の符号に従う。

- 変位$u(x,t)$が近傍点の変位の平均より小さければ、xは上向きの加速度$u_{tt} \geq 0$を得る（上向きの力をうける）
- 変位$u(x,t)$が近傍点の変位の平均に等しければ、xの加速度$u_{tt} = 0$（力を受けない）
- 変位$u(x,t)$が近傍点の変位の平均より大きければ、xは下向きの加速度$u_{tt} \leq 0$を得る（下向きの力をうける）

2次元での $\frac{\partial^2 u}{\partial x^2}$



(a)頂部では $\nabla^2 u = u_{xx} + u_{yy} \le 0$

(b)底部では $\nabla^2 u = u_{xx} + u_{yy} \ge 0$

2.3 楕円型(Elliptic type) の差分化

Laplace 方程式

$$\nabla^2 u = 0$$

Poisson 方程式

$$\nabla^2 u = f$$

$\nabla^2 u = -\rho$: 静電場ポテンシャル

$\nabla^2 T = -g(x, y)$: 定常状態温度分布（$g(x, y) > 0$ 熱発生，$g(x, y) > 0$ 熱吸収）

Helmholtz 方程式

$\nabla^2 u + \lambda u = 0$: 太鼓の膜の振動モード