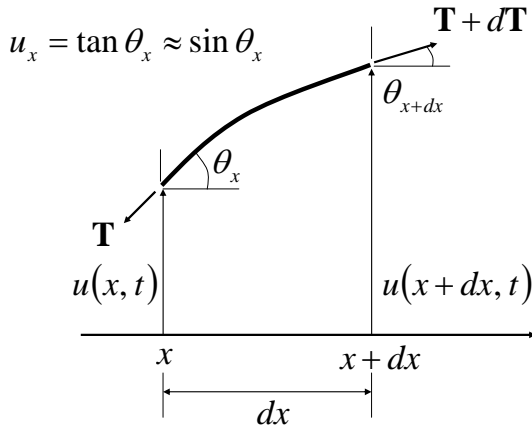


3.微分方程式の導出

3.1 一次元波動方程式の導入(Introduction to Wave Propagation in 1D)



振動する弦の小片 ($x, x + \Delta x$) と作用する力
 Infinitesimal element ($x, x + \Delta x$) of bowstring in 1D
 vibration and forces T (internal tension) applied to it.

- 微分表記 (Differentiations for differential)

$$u = u(x, t), \quad u_x = \frac{\partial u}{\partial x}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}$$

- 微小振幅 (Infinitesimal Amplitude)

$$\theta \rightarrow 0 \quad \cos \theta \rightarrow 1, \quad \sin \theta \rightarrow \theta, \quad \tan \theta \rightarrow \theta: \quad \theta_x \approx \left. \frac{\partial u}{\partial x} \right|_x$$

- 釣り合い式・運動方程式 (Equilibrium / Motion Equation)

$$-T \cos \theta_x + (T + dT) \cos \theta_{x+dx} = 0 \quad x\text{-direction}$$

$$-\rho \cdot dx \cdot u_{tt} - T \sin \theta_x + (T + dT) \sin \theta_{x+dx} = 0 \quad y\text{-direction}$$

$$dT \approx 0 \quad x\text{-direction}$$

$$-\rho \cdot dx \cdot u_{tt} - T(u_x - u_{x+dx}) = 0 \quad y\text{-direction}$$

$$T(x, t) = T(x + dx, t) \quad x\text{-direction}$$

$$u_{tt} = \frac{T}{\rho} u_{xx} = \left(\frac{T}{\rho} \frac{u_{x+dx} - u_x}{dx} \right)_{dx \rightarrow 0} \quad y\text{-direction}$$

3. 微分方程式の解析解: 解放

3.1 波動方程式の D'Alembert 解 (D'Alembert's Solution for Differential Equations for Wave Propagation)

Differential Equation

$$u_{tt} = V^2 u_{xx}, \quad (-\infty < x < \infty, \quad 0 < t < \infty) \quad (1)$$

Initial Conditions: We must two initial conditions to solve wave propagation with second differential equation

$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}, \quad (-\infty < x < \infty) \quad (2)$$

D'Alembert's Solution

1.- Replacement: $\{x, t\} \Rightarrow \{\xi, \eta\}$

$$\begin{cases} \xi = x + Vt \\ \eta = x - Vt \end{cases} \quad (3)$$

$$u_{\xi\eta} = 0 \quad (4)$$

$$\begin{cases} u_x = u_\xi + u_\eta \\ u_t = V(u_\xi - u_\eta) \\ u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{tt} = V^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \end{cases} \quad (5)$$

2.- Integration

$$u_\eta(\xi, \eta) = \phi(\eta); \text{ Arbitrary Function } \phi(\eta) \text{ as } \eta \quad (6)$$

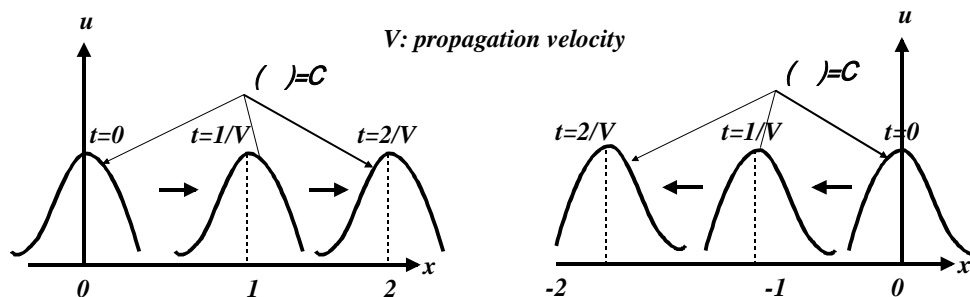
$$u(\xi, \eta) = \Phi(\eta) + \theta(\xi) \quad (7)$$

Arbitrary Function $\theta(\xi)$ as ξ

$$\Phi(\eta) = \int \phi(\eta) d\eta$$

3.- Replacement: $\{\xi, \eta\} \Rightarrow \{x, t\}$

$$u(x, t) = \phi(x - Vt) + \theta(x + Vt) \tag{8}$$



4.- Initial Condition:

$$u(x, t) = \phi(x - Vt) + \theta(x + Vt)$$

$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$

$$\begin{cases} \phi(x) + \theta(x) = f(x) \\ -V\phi'(x) + V\theta'(x) = g(x) \end{cases} \tag{9}$$

$$-V\phi(x) + V\theta(x) = \int_{x_0}^x g(\xi) d\xi + K \tag{10}$$

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2V} \int_{x_0}^x g(\xi) d\xi - K \tag{11}$$

$$\theta(x) = \frac{1}{2} f(x) + \frac{1}{2V} \int_{x_0}^x g(\xi) d\xi + K \tag{12}$$

$$u(x, t) = \frac{1}{2} [f(x - Vt) + f(x + Vt)] + \frac{1}{2V} \int_{x-Vt}^{x+Vt} g(\xi) d\xi \tag{13}$$

- 特性曲線(Characteristic Curve)

Example-1

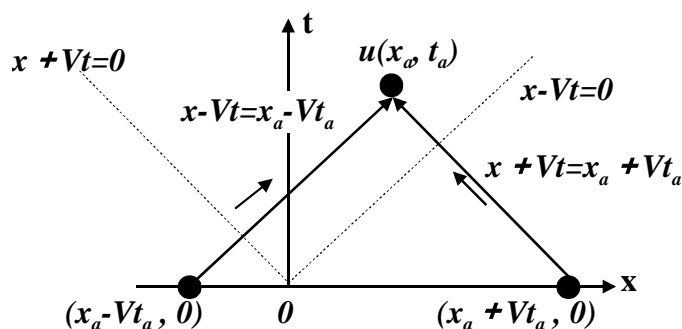
Initial condition

$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) = 0 \end{cases} \quad (14)$$

$$u(x, t) = \frac{1}{2} [f(x - Vt) + f(x + Vt)] \quad (15)$$

characteristic curve

$$\begin{cases} x - Vt = x_a - Vt_a \\ x + Vt = x_a + Vt_a \end{cases} \quad (16)$$



Example-2

$$\begin{cases} u(x, 0) = f(x) = \begin{cases} 1 & (-1 \leq x \leq 1) \\ 0 & \text{the other points} \end{cases} \\ u_t(x, 0) = 0 \end{cases} \quad (17)$$

Characteristic curve

$$x = -1 \text{ or } 1, t = 0$$

$$\begin{cases} x - Vt = \pm 1 \\ x + Vt = \pm 1 \end{cases} \quad (18)$$

