

## フーリエ変換と地震のスペクトル分析

## Spectrum Analysis of Earthquake and Fourier Transform

\* このプリントは「新・地震動のスペクトル解析入門」（大崎順彦, 鹿島出版会）を元に作成しています。: The original of this document was from the above text book.

- 世界最初の強震記録：強震計（Strong-Motion Acceleration: SMAC）

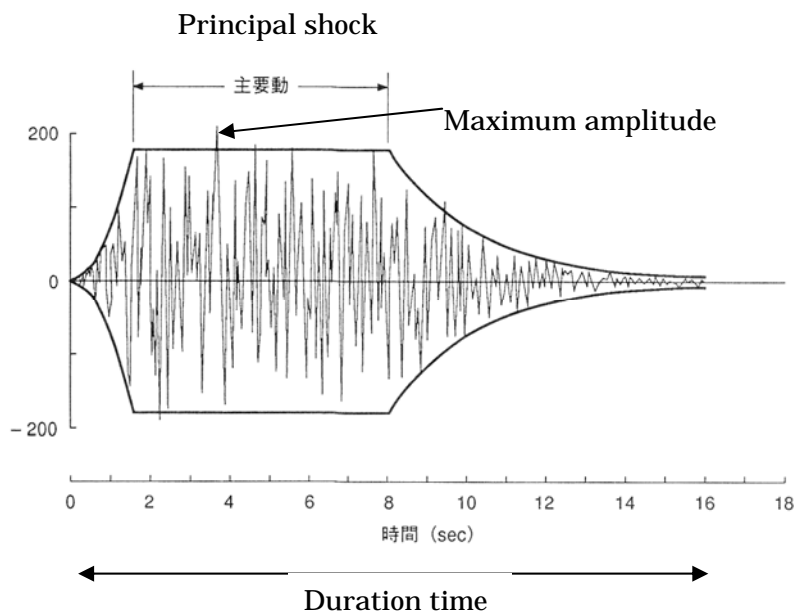
・ Imperial Valley Earthquake at EL Centro(1940.5.18) 326gal：人類初の強震の記録  
(Record of the first strong motion due to earthquake in the human history)

・ Arvin-Tahachapi Earthquake at Taft(1952.7.12) 147gal

- 地震波の特長（Properties of Earthquake Wave）

対象とする地盤 - 構造物系に対して与える影響を考えるための有力な手がかり。

- ・ 最大振幅（maximum amplitude）
- ・ 継続時間（duration time）
- ・ 包絡曲線（envelope curve）：主要動（principal shock）
- ・ 波数（numbers of wave）
- ・ 振動周期（periods）
- ・ エネルギー（Energy）



## Finite Fourier Approximation of Time History and Time Series and its Formulations

### 1) Approximation of digital time history data with Tri-angle series

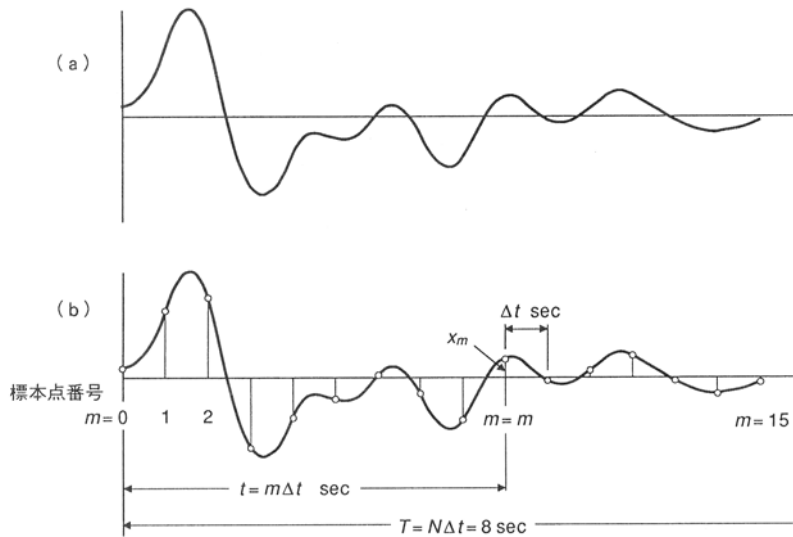
Discrete System:  $\Delta t > 0$  : data sampling interval

$(t_0, x_0), (t_1, x_1), (t_2, x_2), \dots, (t_{N-1}, x_{N-1}) \dots N=(N-1)+1$ : data: *N conditions*

Duration Time:  $T = N \Delta t$  (1.1)

Time:  $t = m \Delta t (m=0, 1, 2, \dots, N-1)$  (1.2)

A data point:  $x_m = x(m \Delta t)$



$N=16(m=0 - 15) : x_0, x_1, \dots, x_{15}$ .  
duration time:  $T=16 \times \Delta t$

### 2) Approximation of digital time history data with infinite tri-angle series

$$\left. \begin{aligned} &A_0 + A_1 \cos t + A_2 \cos 2t + \dots + A_k \cos kt + \dots \\ &+ B_0 + B_1 \sin t + B_2 \sin 2t + \dots + B_k \sin kt + \dots \end{aligned} \right\} \quad (2.1)$$

What is the period  $T_p$  of  $\cos(kt)$  or  $\sin(kt)$ ?

$$\cos kt = \cos k(t + T_p) = \cos(kt + 2\pi) = \cos k \left( t + \frac{2\pi}{k} \right)$$

$\therefore T_p = \frac{2\pi}{k}$  : As  $k$  increases, the period  $T_p$  decrease  
(the frequency  $f=1/T_p$  increases).

$$\sum_{k=0}^{\infty} [A_k \cos kt + B_k \sin kt] \quad (2.2)$$

replace  $t$  by  $\frac{2\pi}{T}t$  or  $\frac{2\pi}{N\Delta t}t$

$$\sum_{k=0}^{\infty} \left[ A_k \cos \frac{2\pi kt}{N\Delta t} + B_k \sin \frac{2\pi kt}{N\Delta t} \right] \quad (2.3)$$

**3) Approximation of digital time history data with finite triangle series**

**Set  $k$  to be from 0 to  $N/2$**

$$x_m = \sum_{k=0}^{N/2} \left[ A_k \cos \frac{2\pi kt}{N\Delta t} + B_k \sin \frac{2\pi kt}{N\Delta t} \right] = \sum_{k=0}^{N/2} \left[ A_k \cos \frac{2\pi km}{N} + B_k \sin \frac{2\pi km}{N} \right] \quad (3.1)$$

$$\left. \begin{matrix} A_0, A_1, A_2, \dots, A_k, \dots, A_{N/2} \\ B_0, B_1, B_2, \dots, B_k, \dots, B_{N/2} \end{matrix} \right\} \text{ Here, Number of unknown coefficients is } 2(N/2+1) \quad (3.2)$$

Number of unknown coefficient  $2(N/2+1) = N+2 >$  number of conditions (data)  $N$

**From partial consideration,**

For the case of  $k = 0$ ,  $A_0 \cos \frac{2\pi km}{N} = A_0 \cdot 1 = A_0$  and  $B_0 \sin \frac{2\pi km}{N} = B_0 \cdot 0 \equiv 0$  (3.3)

For the case of  $k = N/2$ ,  $B_{N/2} \sin \frac{2\pi km}{N} = B_{N/2} \sin \pi m \equiv 0$  (3.4)

**Consequently, Eq.(3.1) is reduced to**

$$x_m = A_0 + \sum_{k=1}^{N/2-1} \left[ A_k \cos \frac{2\pi km}{N} + B_k \sin \frac{2\pi km}{N} \right] + A_{N/2} \cos \frac{2\pi(N/2)m}{N} \quad (3.5)$$

For convenience

$$x_m = A_0/2 + \sum_{k=1}^{N/2-1} \left[ A_k \cos \frac{2\pi km}{N} + B_k \sin \frac{2\pi km}{N} \right] + A_{N/2}/2 \cos \frac{2\pi(N/2)m}{N} \quad (3.6)$$

$$\left. \begin{matrix} A_0, A_1, A_2, \dots, A_k, \dots, A_{N/2-1}, A_{N/2} \\ B_1, B_2, \dots, B_k, \dots, B_{N/2-1} \end{matrix} \right\} N/2+1+N/2-1=N \quad (3.7)$$

**Therefore,**

Numbers of unknown coefficient  $N =$  Condition Equation  $N$  (3.8)

**4) Determination of  $A_k$  and  $B_k$  with orthogonal property of triangle functions**

**なぜ三角関数か？ / Why do we use triangle functions for Fourier approximation?**

**三角関数系の直交性を利用する (We utilize orthogonal property for Triangle Functions)**

$$2 \cos \alpha \cdot \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta) \quad (a)$$

$$2 \cos \alpha \cdot \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta) \quad (b)$$

$$2 \sin \alpha \cdot \sin \beta = -\cos(\alpha + \beta) + \cos(\alpha - \beta) \quad (c)$$

$$2 \cos^2 \alpha = 1 + \cos 2\alpha \quad (d)$$

$$2 \sin^2 \alpha = 1 - \cos 2\alpha \quad (e)$$

$$\begin{aligned} \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos\{\alpha + (N-1)\beta\} \\ = \frac{\cos\left(\alpha + \frac{N-1}{2}\beta\right) \sin \frac{N\beta}{2}}{\sin \frac{\beta}{2}} \end{aligned} \quad (f)$$

$$\begin{aligned} \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin\{\alpha + (N-1)\beta\} \\ = \frac{\sin\left(\alpha + \frac{N-1}{2}\beta\right) \sin \frac{N\beta}{2}}{\sin \frac{\beta}{2}} \end{aligned} \quad (g)$$

if  $\alpha = 0$ , summarize the results.

$$\sum_{m=0}^{N-1} \cos \beta m = \frac{\cos \frac{N-1}{2} \beta \cdot \sin \frac{N\beta}{2}}{\sin \frac{\beta}{2}} \quad (h)$$

$$\sum_{m=0}^{N-1} \sin \beta m = \frac{\sin \frac{N-1}{2} \beta \cdot \sin \frac{N\beta}{2}}{\sin \frac{\beta}{2}} \quad (i)$$

$$\left. \begin{aligned} \sum_{m=0}^{N-1} \cos \frac{2\pi l m}{N} \cos \frac{2\pi k m}{N} &= \begin{cases} N/2 & k=l \\ 0 & k \neq l \end{cases} \\ \sum_{m=0}^{N-1} \sin \frac{2\pi l m}{N} \sin \frac{2\pi k m}{N} &= \begin{cases} N/2 & k=l \\ 0 & k \neq l \end{cases} \\ \sum_{m=0}^{N-1} \sin \frac{2\pi l m}{N} \cos \frac{2\pi k m}{N} &= 0 \end{aligned} \right\} \quad (j)$$

For  $A_k$

たとえば、 $A_k$  を求める。 / For example, we calculate the factor  $A_k$ .

$$x_m = A_0/2 + \sum_{l=1}^{N/2-1} \left[ A_l \cos \frac{2\pi lm}{N} + B_l \sin \frac{2\pi lm}{N} \right] + A_{N/2}/2 \cos \frac{2\pi(N/2)m}{N} \quad (4.1)$$

1) 上式の両辺に  $\cos(2\pi km/N)$  を掛ける / Multiplication of  $\cos(2\pi km/N)$  to Eq.(4.1).

$$\begin{aligned} x_m \cos(2\pi km/N) &= \frac{A_0}{2} \cos \frac{2\pi km}{N} \\ &+ \sum_{l=1}^{N/2-1} \left[ A_l \cos \frac{2\pi lm}{N} \cos \frac{2\pi km}{N} + B_l \sin \frac{2\pi lm}{N} \cos \frac{2\pi km}{N} \right] \\ &+ \frac{A_{N/2}}{2} \cos \frac{2\pi(N/2)m}{N} \cos \frac{2\pi km}{N} \end{aligned} \quad (4.2)$$

2)  $m=0$  から  $m=N-1$  まで総和をとる / Summation from  $m=0$  to  $m=N-1$  in Eq. (4.2)

$$\begin{aligned} \sum_{m=0}^{N-1} x_m \cos(2\pi km/N) &= \frac{A_0}{2} \sum_{m=0}^{N-1} \cos \frac{2\pi km}{N} \quad (-> 0) \\ &+ \sum_{l=1}^{N/2-1} \left[ \sum_{m=0}^{N-1} A_l \cos \frac{2\pi lm}{N} \cos \frac{2\pi km}{N} \right] \\ &+ \sum_{l=1}^{N/2-1} \left[ \sum_{m=0}^{N-1} B_l \sin \frac{2\pi lm}{N} \cos \frac{2\pi km}{N} \right] \quad (-> 0) \\ &+ \frac{A_{N/2}}{2} \sum_{m=0}^{N-1} \cos \frac{2\pi(N/2)m}{N} \cos \frac{2\pi km}{N} \quad (-> 0) \end{aligned} \quad (4.3)$$

1st term, 3rd term and 4th term in right formula =0 with account for the orthogonal

$$\sum_{m=0}^{N-1} x_m \cos(2\pi km/N) = \sum_{l=1}^{N/2-1} A_l \left[ \sum_{m=0}^{N-1} \cos \frac{2\pi lm}{N} \cos \frac{2\pi km}{N} \right] \quad (4.4)$$

$$\sum_{l=1}^{N/2-1} A_l \left[ \sum_{m=0}^{N-1} \cos \frac{2\pi lm}{N} \cos \frac{2\pi km}{N} \right] = A_1 \cdot 0 + A_2 \cdot 0 + \dots + A_k \cdot \frac{N}{2} + \dots + A_{N/2-1} \cdot 0 \quad (4.5)$$

$$A_k = \frac{2}{N} \sum_{m=1}^{N-1} x_m \cos \frac{2\pi km}{N} \quad (4.6)$$

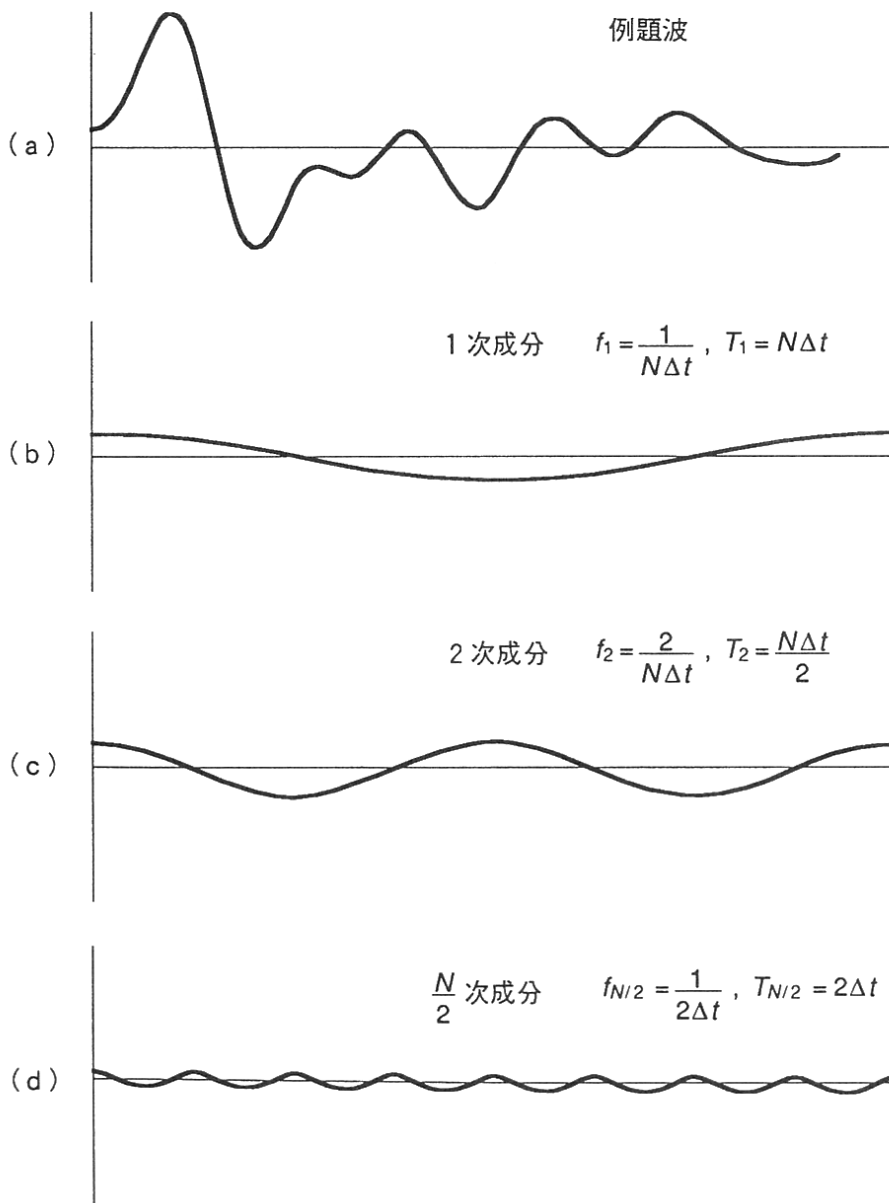
$$\left. \begin{aligned} A_k &= \frac{2}{N} \sum_{m=1}^{N-1} x_m \cos \frac{2\pi km}{N} \\ B_k &= \frac{2}{N} \sum_{m=1}^{N-1} x_m \sin \frac{2\pi km}{N} \end{aligned} \right\} \begin{aligned} &k = 0, 1, 2, \dots, N/2-1, N/2 \\ &k = 1, 2, \dots, N/2-1 \end{aligned} \quad (4.7)$$

$$\frac{A_0}{2} = \frac{1}{N} \sum_{m=0}^{N-1} x_m : \text{mean value} \quad (4.8)$$

3)時間関数  $x(t)$  の近似: *Fourier Approximation*

$$t = m\Delta t, \quad m = \frac{t}{\Delta t} \tag{4.9}$$

$$x(t) \approx A_0/2 + \sum_{k=1}^{N/2-1} \left[ A_k \cos \frac{2\pi kt}{N \Delta t} + B_k \sin \frac{2\pi kt}{N \Delta t} \right] + A_{N/2}/2 \cos \frac{2\pi(N/2)t}{N \Delta t} \tag{4.10}$$



**5) Spectrum Properties**

$$x(t) \approx A_0/2 + \sum_{k=1}^{N/2-1} \left[ A_k \cos \frac{2\pi kt}{N \Delta t} + B_k \sin \frac{2\pi kt}{N \Delta t} \right] + A_{N/2}/2 \cos \frac{2\pi(N/2)t}{N \Delta t} \quad (5.1)$$

1) 周波数・周期 (Frequency/Period)

$$\omega_k = \frac{2\pi}{N\Delta t} k = 2\pi \frac{k}{T} \quad (5.2)$$

$$T_k = \frac{2\pi}{\omega_k} = \frac{T}{k} = \frac{N\Delta T}{k}, \quad f_k = \frac{1}{T_k} = \frac{k}{T} = \frac{k}{N\Delta T} \quad (5.3)$$

-  $k = 0, f_k = f_0 = 0$ : 直流成分(Cascade Component)

$$\frac{A_0}{2} = \frac{1}{N} \sum_{m=0}^{N-1} x_m \quad \text{全体のゼロ点からのズレ} \quad (5.4)$$

-  $k \neq 0, f_k \neq 0$ :

$$f_1 < f_2 < \dots < f_{N/2-1} < f_{N/2}, \quad T_1 > T_2 > \dots > T_{N/2-1} > T_{N/2} \quad (5.5)$$

- 分解する周波数はトビトビ (Discontinuity of Decomposed Frequency)

$$\Delta f \equiv f_{k+1} - f_k = \frac{1}{N\Delta t} \quad (5.6)$$

2) 基本振動数(Fundamental Frequency)

$$f_1 = \frac{1}{T_1} = \frac{1}{T} = \frac{1}{N\Delta T} \quad (5.7)$$

3) ナイキスト振動数(Nyquist Frequency) 分解能: Resolving power

検出可能な高周波数の限界値  $f$ : Limit value of detection possible high frequency

$$f_{N/2} = \frac{1}{T_{N/2}} = \frac{1}{2\Delta T} \quad (5.8)$$

$$\Delta t = 0.01 \text{ (sec.)} \rightarrow f_{N/2} = \frac{1}{2 \cdot 0.01} = 50 \text{ Hz} \quad (5.9)$$

4) 振幅・位相角(Amplitude/Phase Angle)

情報は2つ

$$A_k \cos(\omega t) + B_k \sin(\omega t) = X_k \cos(\omega t + \phi) \quad (5.10)$$

$$X_k = \sqrt{A_k^2 + B_k^2} \quad (5.11)$$

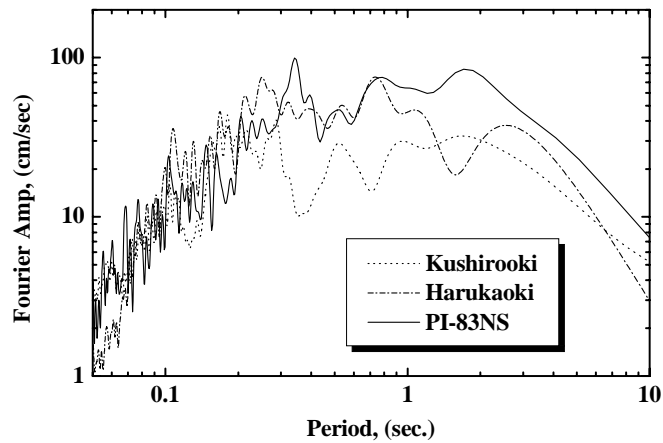
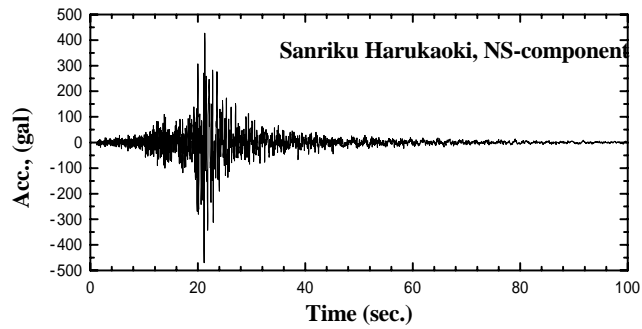
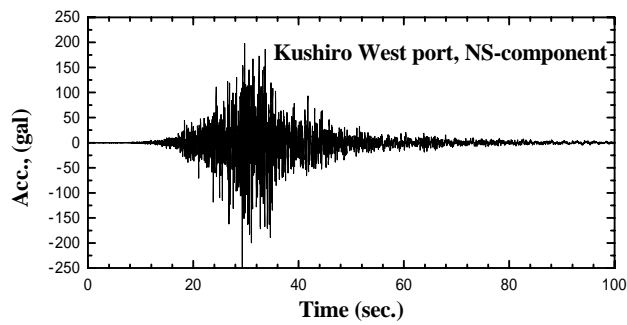
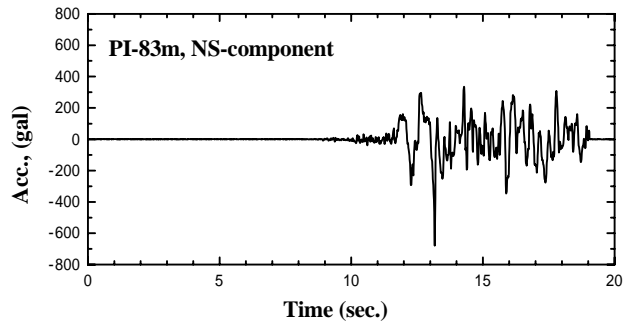
$$\phi = \tan^{-1} \left( -\frac{B_k}{A_k} \right) \quad (5.12)$$

4) フーリエ (振幅) スペクトル(Fourier Amplitude Spectrum/ Fourier Spectrum)

$$\frac{T}{2} X_k \quad \text{dimension: } [X_k] [\text{sec.}] \quad (5.13)$$

5) パワースペクトル(Power Spectrum): Invariant Value

$$\sum_{m=0}^{N-1} x_m^2 \Delta t = T |C_0|^2 + 2 \sum_{k=1}^{N/2-1} T |C_k|^2 + T |C_{N/2}|^2 \quad C_k: \text{複素数フーリエ振幅} \quad (5.14)$$





**6) Finite Fourier Approximation with Complex Number**

$$c = a + ib : c : \text{complex number, } a : \text{real part, } b : \text{imaginary part, } i = \sqrt{-1} \quad (6.1)$$

$$|c| = \sqrt{a^2 + b^2} : \text{absolute value} \quad (6.2)$$

$$c \cdot c^* = |c|^2, \quad c^* = a - ib : \text{conjugate complex number} \quad (6.3)$$

$$e^{\pm i\theta} = (\cos \theta \pm i \sin \theta) : \text{Euler's Formula} \quad (6.4)$$

$$\left. \begin{aligned} \cos \theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin \theta &= \frac{1}{2}(e^{i\theta} - e^{-i\theta}) \end{aligned} \right\} \quad (6.5)$$

**Approximation with Complex Number**

$$\left. \begin{aligned} \cos \frac{2\pi km}{N} &= \frac{1}{2} [e^{i(2\pi km/N)} + e^{-i(2\pi km/N)}] \\ \sin \frac{2\pi km}{N} &= -i \frac{1}{2} [e^{i(2\pi km/N)} - e^{-i(2\pi km/N)}] \end{aligned} \right\} \quad (6.6)$$

**Finite series**

$$x_m = \sum_{k=0}^{N-1} C_k e^{i \frac{2\pi km}{N}}, \quad m = 0, 1, 2, \dots, N-1 \quad (6.7)$$

$$C_k = \frac{A_k - iB_k}{2}, \quad k = 0, 1, 2, \dots, N-1 \quad N \quad (6.8)$$

**Determination of  $C_k$**

$$C_k = \frac{1}{N} \sum_{m=0}^{N-1} x_m e^{-i \frac{2\pi km}{N}}, \quad k = 0, 1, 2, \dots, N-1 \quad N \quad (6.9)$$

$$C_k = C_{N-k} : \text{folding frequency } f_{N/2} = \frac{1}{2\Delta t} \quad (6.10)$$

$$\left. \begin{aligned} A_k &= 2 \text{Real}(C_k) \\ B_k &= 2 \text{Im}(C_k) \end{aligned} \right\}, \quad k = 0, 1, 2, \dots, N/2 \quad (6.11)$$

**7) Fast Fourier Transform ( FFT )**

	$C_0 \ C_1 \ C_2 \ C_3 \ C_4 \ C_5 \ C_6 \ C_7$	$1 \times \underline{8}$
一回分割	$C_0 \ C_2 \ C_4 \ C_6$ $C_1 \ C_3 \ C_5 \ C_7$	$2 \times \underline{4}$
2回分割	$C_0 \ C_4$ $C_2 \ C_6$ $C_1 \ C_5$ $C_3 \ C_7$	$4 \times \underline{2}$
3回分割	$C_0$ $C_4$ $C_2$ $C_6$ $C_1$ $C_5$ $C_3$ $C_7$	$8 \times \underline{1}$

- 計算時間(Time for Fourier Coefficient Calculations):  $T_{cal}$

Fourier Transform(FT)  $T_{cal} \propto N^2$

Fast Fourier Transform(FFT)  $T_{cal} \propto N \log_2 N$

<b>Comparison for Cal. Time</b>		
<b>N</b>	<b>Factor</b>	<b>Ratio for <math>T_{cal}</math></b>
4094	$2 \times 23 \times 89$	12.9
4095	$3^2 \times 5 \times 7 \times 13$	3.9
4096	$2^{12}$	1
4097	$17 \times 241$	28
4098	$2 \times 3 \times 683$	77
4099	-	460
4100	$2^2 \times 5^2 \times 41$	6.3

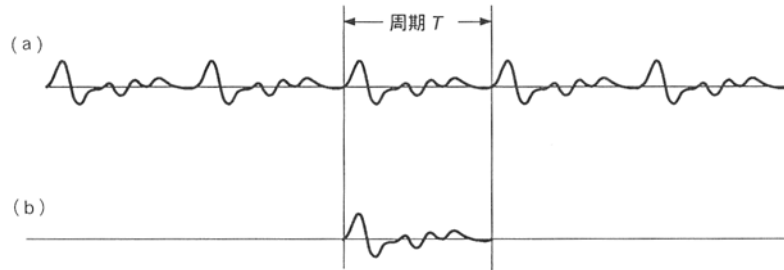
- 後続のゼロ(Trailing Zero):

- N が 2 累乗でないとき、ゼロを後続させる。

$$N=3000 \rightarrow N=3000+1096=4096=2^{12}$$

- リンク効果(Link Effect)を解消

**8) Link effect**



(a) Periodic Function: earthquake motion transformed by Fourier series  
(b) Non-periodic function: real earthquake motion

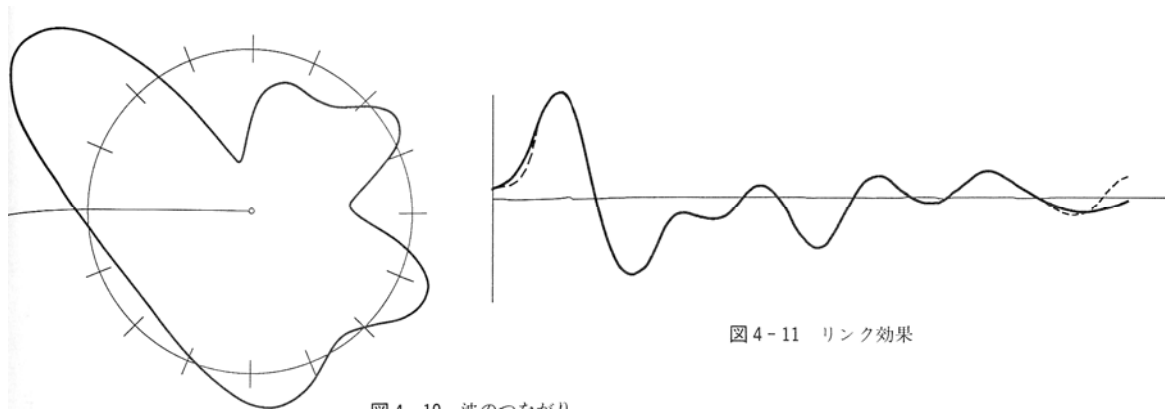
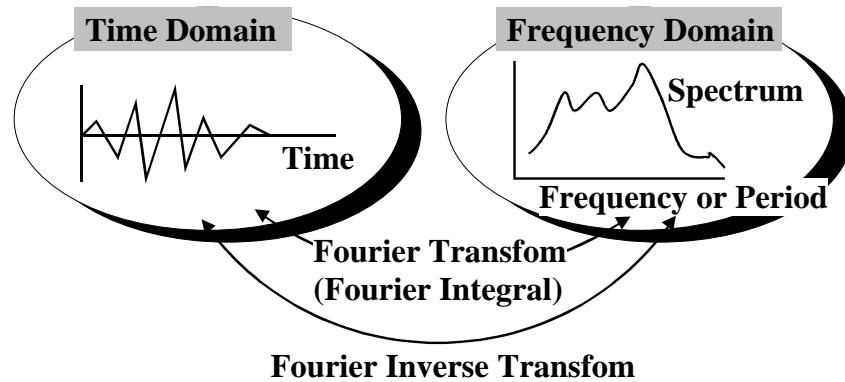


図 4 - 11 リンク効果

図 4 - 10 波のつながり

Link Effect in Fourier transform

**9) Fourier Integral: Discrete system / Continuous System**



$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{i\frac{2\pi kt}{T}} = \sum_{k=-\infty}^{\infty} (TC_k) e^{i\frac{2\pi kt}{T}} \frac{1}{T} \quad \text{: for Discrete System} \quad (9.1)$$

$$C_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-i\frac{2\pi kt}{T}} dt, \quad -\infty \leq k \leq \infty \quad (9.2)$$

$$f_k = \frac{k}{T}, \quad \Delta f = f_{k+1} - f_k = \frac{1}{T} \quad (9.3)$$

$$T \rightarrow \infty \left\{ \begin{array}{l} \frac{k}{T} \rightarrow f \\ \Delta f = \frac{1}{T} \rightarrow df \approx 0 \\ TC_k : \text{discrete} \rightarrow F(f) : \text{continuous function} \end{array} \right. \quad (9.4)$$

$$T \rightarrow \infty \left\{ \begin{array}{l} \frac{k}{T} \rightarrow f \\ \Delta f = \frac{1}{T} \rightarrow df \approx 0 \\ TC_k : \text{discrete} \rightarrow F(f) : \text{continuous function} \\ x(t) = \sum_{k=-\infty}^{\infty} (TC_k) e^{i\frac{2\pi kt}{T}} \frac{1}{T} \rightarrow x(t) = \int_{-\infty}^{\infty} F(f) e^{i2\pi ft} df \\ TC_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-i\frac{2\pi kt}{T}} dt \rightarrow F(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt \end{array} \right. \quad (9.5)$$

**Fourier transform(Fourier integral):**  $x(t) \rightarrow F(f) = \lim_{T \rightarrow \infty} (TC_k)$  (9.6)

**Fourier inverse transform:**  $F(f) \rightarrow x(t)$  (9.7)

**Fourier Spectrum:**  $T|C_k|$

$$TC_k = \frac{T}{2}(A_k - iB_k) \quad (9.8)$$

$$T|C_k| = \frac{T}{2} \sqrt{A_k^2 + B_k^2} = \frac{T}{2} X_k \quad (9.9)$$

**10) Smoothing / Filters**

**a) Data Window**

**b) Spectral Window**

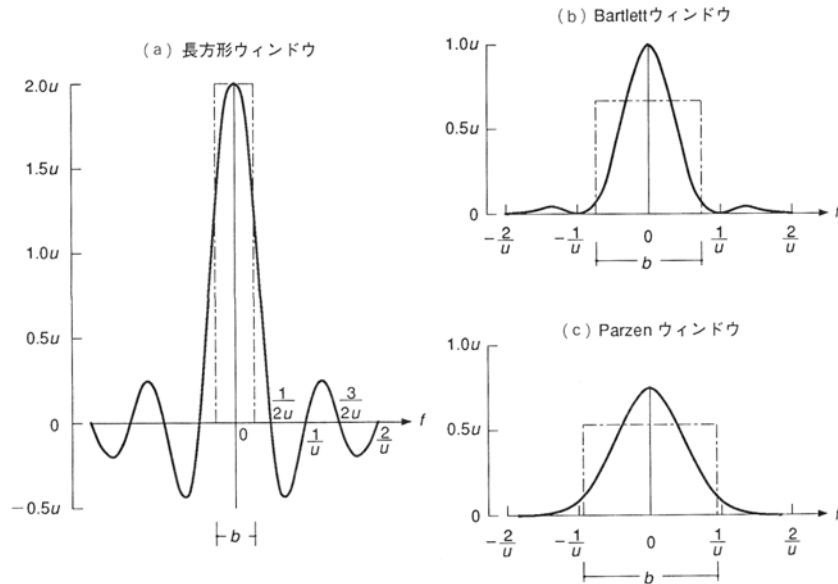
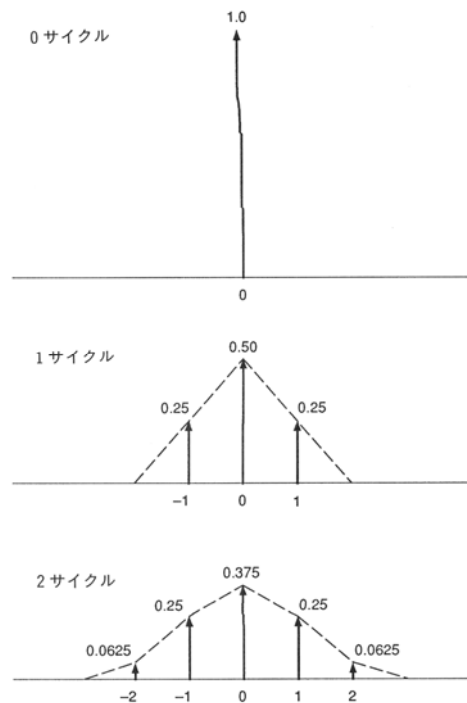
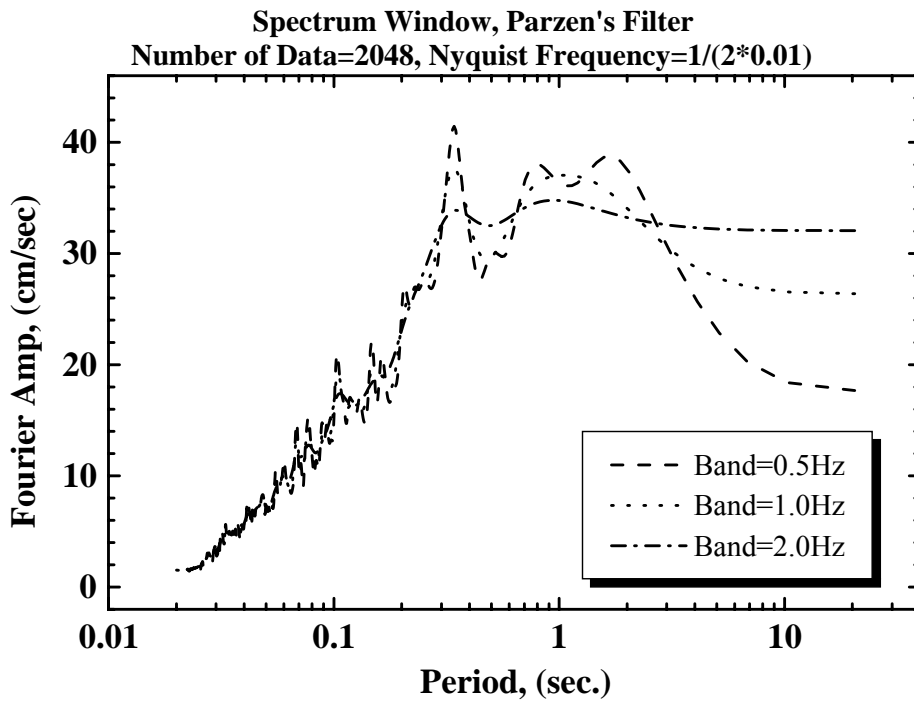
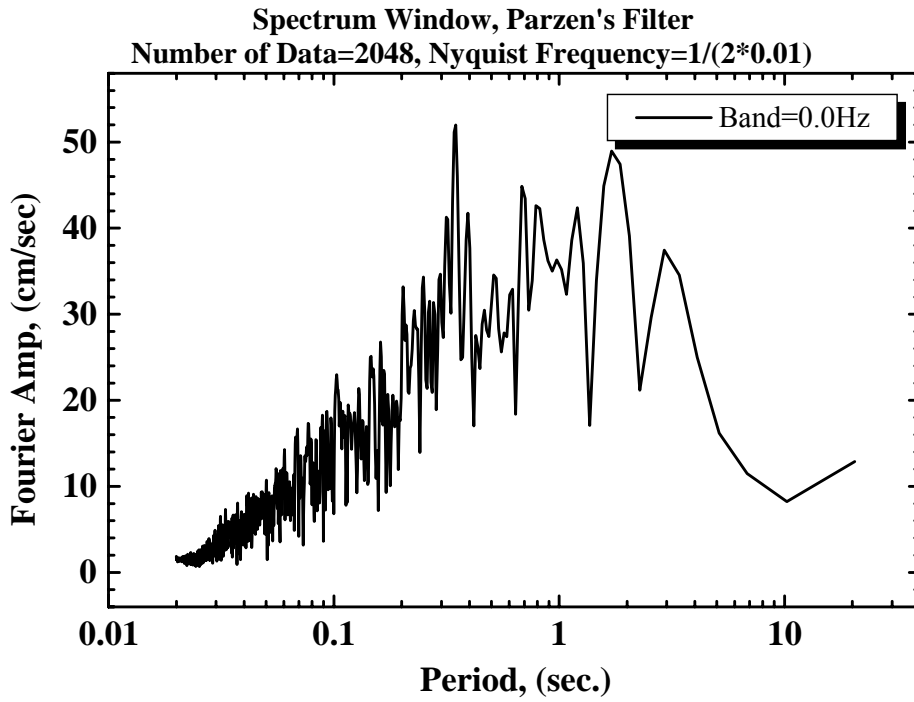
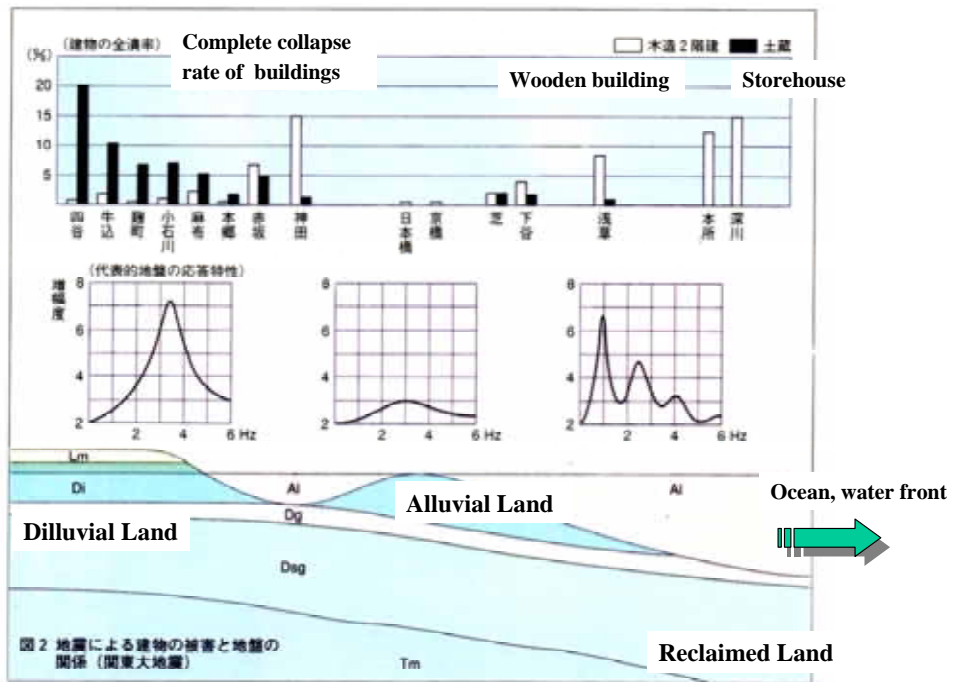


図6-6 スペクトル・ウィンドウ

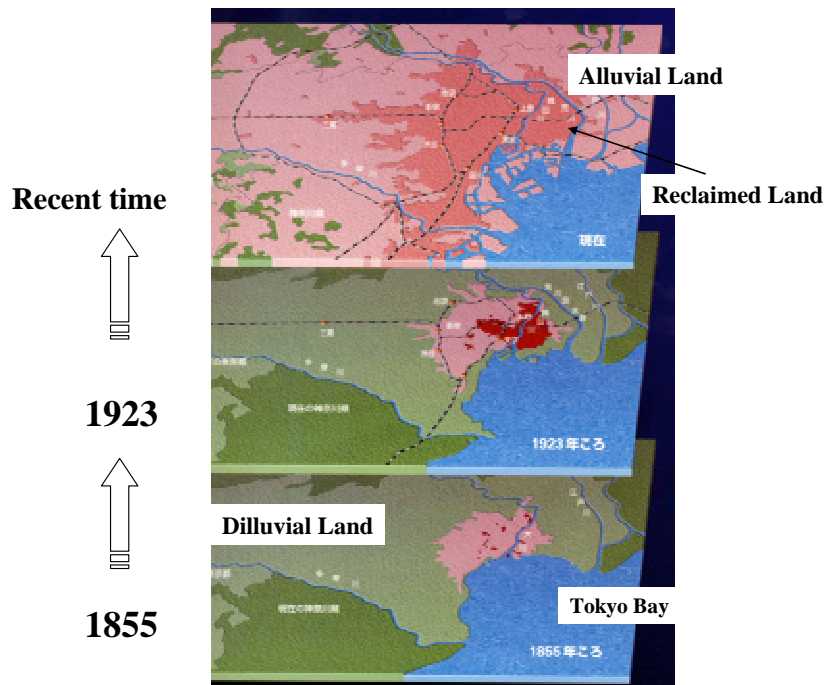


**c) Lag Window**





地震による建物の被害と地盤の関係: Relation between the damage of the building by the earthquake and ground condition: 「おもしろジオテク」(技報堂)



東京のウォーター・フロントの経緯(埋立て事業の経緯): Process of waterfront and Reclamation in Tokyo-bay

推薦図書 (Recommendation Text and Papers)

- 1) 「新・地震動のスペクトル解析入門」 大崎順彦, 鹿島出版会
- 2) 「スペクトル解析」 日野幹雄, 朝倉書店
- 3) 「フーリエ解析」 大石進一, 岩波書店 (理工系の数学入門コース)

For examples

- 1) “The Fourier Integral and Its Applications”, papoulis, A.(1962), McGraw-Hill
- 2) “Random Data: Analysis and measurement Procedures”, Bendat, J.S. and Piersol, A.G.(1971), John Wiley & Sons.